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NONLINEAR OSCILLATIONS OF A BUBBLE UNDER VARIOUS BOUNDARY CONDITIONS

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The method of many scales is used to examine the nonlinear oscillations of a spherical gas bubble that occur under the action of a periodically changing external pressure in a spherical volume of inviscid, incompressible liquid and in a liquid flow. The influence of the finite dimensions of the volume and liquid flow velocity on the conditions for the existence of certain types of the equilibrium state of an oscillating bubble is analyzed.

Nonlinear oscillations of gas bubbles are observed in various situations, particularly when an acoustic field propagates through a bubble mixture or a liquid flows past various obstacles. Therefore, there is an obvious need for a theory of nonlinear oscillations of a bubble. The current theory gives the best description of the dynamics of a radially pulsing single bubble in an infinite volume of liquid [1, 2]. In a number of cases, patterns of transition are analyzed: from regular to random vibrations at the initial stage with a change in the external parameters of the effect on a nonlinearly oscillating bubble. Particular attention is paid here to the bifurcation mechanism underlying the change of equilibrium type of a nonlinear system [3, 4]. However, the considered models of nonlinear systems did not analyze the situation when the volume in which a bubble oscillates has finite dimensions or when a radially oscillating bubble itself is subjected additionally to directed external influences. Obviously, the presence of such situations will introduce corrections to the bifurcation mechanism involved in the change of equilibrium type of an oscillating bubble.

It should be noted that in a number of works an analysis of the effect exerted by the shape of the liquid surface on the oscillating gas bubble parameters is performed [5, 6]. But they failed to consider possible types of equilibrium of nonlinear systems and the mechanism of their change. Therefore, we made an attempt to analyze the effect of various boundary conditions on the bifurcation mechanism underlying the change of equilibrium type of an oscillating gas bubble in the region of the main resonance. We considered two situations: a bubble occurring in an acoustic field oscillates radially in a spherical volume of liquid of radius L which greatly exceeds the bubble radius R_0; a bubble oscillates not in a stagnant volume of liquid but in a liquid that moves relative to it with velocity U.

Suppose a spherical bubble oscillates at the center of a spherical liquid volume of size L. Its vibrations are radial and occur in the region of the main resonance with an external periodic influence distributed uniformly over the entire external surface of the sphere volume and directed along its radius. The liquid itself is incompressible and inviscid.

Since the radius of the volume L greatly exceeds of the bubble size $(L/R_0 >> 1)$, we assume that the pressure at the boundary of the spherical layer is determined only by the external periodic influence, which is independent of the character of vibrations of the bubble itself, as if the bubble were in an infinite volume of liquid.

We obtain an equation for the motion of the bubble walls. For this case, we integrate the equation of motion with respect to r within the limits of from $R(\tau)$ to L:

$$\int_R^L \left[\frac{\partial v(r)}{\partial \tau} + \frac{\partial}{\partial r}\frac{v^2(r)}{2} + \frac{\partial}{\partial r}\frac{P}{\rho}\right] dr = 0 . \tag{1}$$

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Assuming that the expression for the velocity v has the form

$$v(r) = \frac{v(R)\, R^2}{r^2}, \quad r \geq R,$$

we obtain the following equation for the change in the bubble radius:

$$\rho \left[R \frac{d^2R}{d\tau^2} - \frac{R}{h} \frac{d^2R}{d\tau^2} + \frac{3}{2} \left(\frac{dR}{d\tau} \right)^2 - \frac{2R}{L} \left(\frac{dR}{d\tau} \right)^2 + \left(\frac{dR}{d\tau} \right) \frac{2R^4}{2L^4} \right] =$$

$$= P(R, \tau) - P(L, \tau). \tag{2}$$

We assume that the pressure on the bubble surface $P(R, \tau)$ is determined by the relation

$$P(R, \tau) = P_0 \left(\frac{R_0}{R} \right)^{3\gamma} - \frac{2\sigma}{R}, \tag{3}$$

where $P_0 = P^*_\infty + 2\sigma/R_0$.

The pressure at the boundary of the spherical layer of radius L is

$$P(L, \tau) = P^*_\infty - P_a \cos \Omega\tau. \tag{4}$$

The amplitude of the bubble oscillations is assumed to the small. Then

$$R = R_0 (1 + x) = R_0 (1 + \varepsilon u). \tag{5}$$

The value of u is of the order of unity, and the perturbation parameter ε lies within the range $0 < \varepsilon \leq 1$.

For further analysis of Eq. (2) it is convenient to use the dimensionless variables

$$t = \Omega_0 \tau, \quad \Omega_0 = \frac{1}{R_0} \left(\frac{P_0}{\rho} \right)^{1/2}, \quad \omega = \frac{\Omega}{\Omega_0}; \quad W = \frac{2\sigma}{R_0 P_0},$$

$$\frac{P_a}{P^*_\infty} = \eta, \quad \frac{P^*_\infty}{P_0} = 1 - W, \quad \frac{P_a}{P_0} = \xi = (1 - W)\eta, \quad l = \frac{L}{R_0}. \tag{6}$$

Substitution of Eqs. (3)-(6) into Eq. (2) yields

$$\varepsilon \ddot{u} (1 + \varepsilon u) \left[1 - \frac{1}{l} (1 + \varepsilon u) \right] + \frac{3}{2} \varepsilon^2 \dot{u}^2 - \frac{2}{l} \varepsilon^2 \dot{u}^2 (1 + \varepsilon u) +$$

$$+ \varepsilon^2 \frac{\dot{u}^2}{2l^4} (1 + \varepsilon u)^4 - (1 + \varepsilon u)^{-3\gamma} + W (1 + \varepsilon u)^{-1} + (1 - W) \times$$

$$\times (1 - \eta \cos \Omega t) = 0, \tag{7}$$

where $\bullet = d/dt$.

Since $x = \varepsilon u$, for the case of the main resonance we assume

$$\xi = \varepsilon^3 P_v, \quad \frac{1}{l} = \varepsilon h, \quad \omega = \omega_0 + \varepsilon^2 \delta, \tag{8}$$

where the parameters P_v, h, and δ are of the order of unity.

With allowance for Eq. (8), relation (7) can be rewritten in the form

$$\ddot{u} + \omega_0^2 u = \varepsilon \left(-\frac{3}{2} \dot{u}^2 + a_1 u^2 - \omega_0^2 uh \right) +$$

$$+ \varepsilon^2 \left(\frac{3}{2} \dot{u}^2 u - a_2 u^3 + P_v \cos \omega t + \frac{1}{2} \dot{u}^2 h + a_1 u^2 h - \omega_0^2 u^2 h - \omega_0^2 uh^2 \right) + \varepsilon^3 \ldots,$$

$$\omega_0^2 = 3\gamma - W, \quad a_1 = \frac{9}{2} \gamma (\gamma + 1) - 2W,$$

$$a_2 = \frac{\gamma}{2} (9\gamma^2 + 18\gamma + 11) - 3W. \tag{9}$$

Using the method of many scales [7, 8] and restricting ourselves to terms of the order of smallness of ε^2, we obtain a system of equations that determines the time dependence of the amplitude of bubble oscillations C and phase shift φ:

$$\frac{dC}{d\tau} = -\frac{\xi}{2\omega_0} \sin \left(\varphi + \frac{\tau}{2l} \right),$$

$$\frac{d\varphi}{d\tau} = \omega_0 - \omega - \frac{C^2 N}{2\omega_0} - \frac{1}{8\omega_0 l^2} + \frac{1}{2l^2} - \frac{\xi}{2\omega_0 C} \cos \left(\varphi + \frac{\tau}{2l} \right), \tag{10}$$

where

$$N = \frac{5a_1}{6\omega_0^2} \left(a_1 - \frac{3}{2} \omega_0^2 \right) - \frac{3}{4} \left(a_2 - \frac{3}{2} \omega_0^2 \right).$$

Knowing the amplitude and the phase of vibrations of the bubble, we can write the equations of motion of the bubble walls:

$$x = C \cos (\omega\tau + \varphi) + C^2 \left[c_1 + c_2 \cos 2 (\omega\tau + \varphi) \right],$$

$$c_1 = \frac{1}{2\omega_0^2} \left(a_1 - \frac{3}{2} \omega_0^2 \right), \quad c_2 = -\frac{1}{6\omega_0^2} \left(a_1 + \frac{3}{2} \omega_0^2 \right). \tag{11}$$

If the vibrations of the bubble are stationary, then $dc/d\tau = 0$, $d\varphi/d\tau = 0$. In this case, equating the left-hand sides of Eqs. (10) to zero, we write an expression for C:

$$C^6 + C^4 \left[\frac{4\omega_0}{N} (\omega - \omega_0) + \frac{1}{2Nl^2} (1 - 4\omega_0) \right] + C^2 \left[\frac{4\omega_0^2}{N^2} (\omega_0 - \omega)^2 + \right.$$

$$\left. + \frac{1}{16N^2 l^4} (1 - 8\omega_0 + 16\omega_0^2) + \frac{\omega_0 (\omega - \omega_0)}{N^2 l^2} (1 + 4\omega_0) \right] - \frac{\xi^2}{N^2} = 0. \tag{12}$$

The phase shift for stationary vibrations is

$$\varphi = -\frac{\tau}{2l}. \tag{13}$$

We will analyze the change of equilibrium type of an oscillating bubble in the region of the main resonance as a function of the amplitude and frequency of the external influence and of the radius of the medium's volume l.

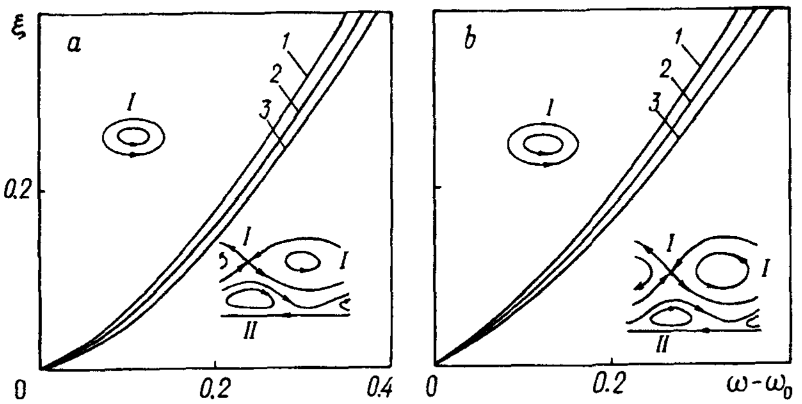

Fig. 1. Bifurcation diagrams and phase trajectories (10) for an oscillating bubble in a spherical volume: a) 1, 2, 3) – (14), $l = 5$, 10, ∞; same (17) in a liquid flow: b) 1, 2, 3) – (20), $V = 0.02$, 0.001, 0; I) with resonance; II) without resonance.

For this purpose, we consider a bubble with initial radius $R_0 = 10^{-5}$ m. Let $\rho = 998$ kg·m^{-3}, $\sigma = 0.0725$ nm^{-1}, $P^*_\infty = 101{,}300$ Pa, and $\gamma = 4/3$.

The results obtained are given in Fig. 1a, where the bifurcation diagram and the corresponding phase trajectories are presented in the coordinate system c, $\varphi + \tau/2l$. The phase trajectories were found as a result of numerical integration of system of equations (10). Curves 1, 2, and 3 were obtained for different values of l; they separate two regions with different types of equilibrium of the nonlinear system. The equation for curves 1, 2, and 3 was derived from Eq. (12):

$$\xi = \frac{2}{3\,(3N)^{1/2}} \left[2\omega_0\,(\omega_0 - \omega) + \frac{4\omega_0 - 1}{4l^2} \right]^{3/2} . \tag{14}$$

From Fig. 1a we can see that with a decrease in ξ we arrive from an upper region with one center at the intersection of the curve described by Eq. (14) at a region with two centers and a saddle, i.e., the change of equilibrium type occurs, which is accompanied by a "saddle-center" bifurcation. Moreover, the smaller l, the higher the amplitude ξ at which the indicated bifurcation is observed for a given frequency of influence.

The vibrations of the bubble in the vicinity of singular points can have resonance and nonresonance characters. The type of vibration is indicated in Fig. 1a.

Thus, if the parameters associated with the external influence are close to the bifurcation line, then the smallest changes in the boundaries of the medium in which vibrations occur can lead to a change in the character of the vibrations of the bubble.

We will consider the second case. Suppose a gas bubble in the field of a periodically varying external pressure oscillates in a liquid flow. The liquid flow moves uniformly with velocity U relative to the bubble. The value of U is rather small. Therefore, we assume that the bubble has a spherical shape and accomplishes radially spherical vibrations. The liquid is incompressible and inviscid. In this case the equation of motion of the bubble walls, with allowance for surface tension, is written as [1]

$$R\,\frac{d^2R}{d^2\tau} + \frac{3}{2}\left(\frac{dR}{d\tau}\right)^2 - \frac{U^2}{4} = \frac{1}{\rho}\,[P\,(R, \tau) - P\,(\infty, \tau)]\,, \tag{15}$$

where $P(R, \tau)$ is defined by relation (3) and the pressure far away from the bubble walls $P(\infty, \tau)$ by relation (4).

Just as in the first case, we introduce dimensionless variables according to Eq. (6). In addition to them, we also use the parameter $V = U^2\rho/P_0$. For the region of the main resonance expression (15) is then rewritten in the following form:

$$\ddot{u} + \omega_0^2 u + \varepsilon \left(-\frac{3}{2} \dot{u}^2 + a_1 u^2 + \frac{\overline{V}}{4} \right) + \varepsilon^2 \left(\frac{3}{2} u \dot{u}^2 - a_2 u^3 - \frac{\overline{V} u}{4} + P_v \cos \omega \tau \right) , \tag{16}$$

where $V = \varepsilon^2 \overline{V}$.

Using the method of many scales and restricting ourselves to terms of the order of smallness of ε^2, we obtain a system of equations for the amplitude and phase of vibrations:

$$\frac{dc}{d\tau} = - \frac{\xi}{2\omega_0} \sin \varphi ,$$

$$\frac{d\varphi}{d\tau} = \omega_0 - \omega - \frac{C^2 N}{2\omega_0} - \frac{V}{\omega_0} \left(\frac{a_1}{2\omega_0^2} - \frac{1}{4} \right) - \frac{\xi}{2\omega_0 C} \cos \varphi . \tag{17}$$

For stationary vibrations of the bubble ($dc/d\tau = d\varphi/d\tau = 0$) Eq. (17) yields expressions for C and φ:

$$C^6 + C^4 \frac{4}{N} \left[\omega_0 (\omega - \omega_0) + 4V \left(\frac{a_1}{2\omega_0^2} - \frac{1}{4} \right) \right] +$$

$$+ C^2 \frac{4\omega_0^2}{N^2} \left[(\omega_0 - \omega)^2 + \frac{V^2}{\omega_0^2} \left(\frac{a_1}{2\omega_0^2} - \frac{1}{4} \right)^2 - \frac{2V}{\omega_0} (\omega_0 - \omega) \left(\frac{a_1}{2\omega_0^2} - \frac{1}{4} \right) \right] - \frac{\xi^2}{N^2} = 0 , \tag{18}$$

$$\varphi = 0 \pm \pi n , \tag{19}$$

where $n = 0, 1, 2, \ldots$.

To analyze the change of equilibrium type of an oscillating bubble as a function of the amplitude and frequency of the external influence and the parameter V, we take the same values of R_0, ρ, σ, P_∞^*, and γ, as in the first case. The results of calculations are given in Fig. 1b, where the bifurcation diagram and corresponding phase trajectories are given in the coordinate system C, φ. Curves 1, 2, and 3 correspond to different values of V. An equation for these curves is obtained from Eq. (18) and has the form

$$\xi = \frac{2}{3 (3N)^{1/2}} \left[2\omega (\omega_0 - \omega) + 2V \left(\frac{a_1}{2\omega_0^2} - \frac{1}{4} \right) \right]^{3/2} . \tag{20}$$

As is seen from Fig. 1b, curves 1, 2, and 3 separate the regions with different types of equilibrium. If for a given frequency of external influence and flow velocity we consider the state of the nonlinear system in the case of a successive decrease in ξ, then from an upper region with one center we arrive at another region with two centers and a saddle. In this case, the change of equilibrium type is accompanied by a 'saddle–center' bifurcation. The larger the value of V, the larger the value of ξ at which the indicated bifurcation is observed for a given frequency.

Curves 3 in Fig. 1b correspond to the limiting cases when, apart from the acoustic field, there are no any other boundary influences, i.e., when $l = \infty$ or $V = 0$. Thus, any of the aforementioned boundary conditions imposed on the known situation in which a gas bubble exposed to a periodically changing external pressure oscillates radially in a stagnant volume of liquid causes a shift to the left of the boundary of the change of equilibrium type of a nonlinear system. In other words, at certain values of the parameters of external influence a nonlinear system is very sensitive to the smallest change in one of these parameters. In this case, a change of equilibrium type is accompanied by a 'saddle–center' bifurcation.

Despite the limitations of the models considered, the above conclusion has a practical importance. First of all, this is due to the fact that in a more real situation the state of a nonlinear system (i.e., the type of equilibrium) can also be determined to a greater extent by the conditions at its boundary.

NOTATION

R, R_0, current and equilibrium radii of a bubble; τ, time; ρ, density of the liquid medium; σ, coefficient of surface tension; Ω, frequency of external influence; ω_0, dimensionless natural frequency of the system; P, pressure in the liquid; P^*_∞, constant pressure; P_a, amplitude of external influence; ω, dimensionless frequency of external influence; γ, polytropic exponent; $v(r)$, $v(R)$ liquid velocity outside the bubble and on its surface.

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